Identification of Characeteristic Velocities of Traffic Systems Using Plant Inversion

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Abstract

Results published in the paper "Dissipation of stop-and-go waves via control of autonomous vehicles" show that it is possible in principle to attenuate the propagation of traffic waves on a ringroad using a single autonomous vehicle. The proposed controller for the vehicle, called the Follower-Stopper, requires prior knowledge of the characteristic velocity of the ringroad in order to fully attenuate the oscillations in system. Here the characeteristic velocity of the vehicle refers to the velocity of each of the vehicles in the constant velocity solution that any traffic dynamical system admits as a solution. Of course in real-world situations one does not have complete knowledge of the state of the system so determining the characteristic velocity poses a challenge. We propose a solution to this problem using plant inversion.

Problem Formulation

Consider a platoon of N vehicles placed on a ring road of length L, where vehicle $i-1$ is ahead of vehicle i. The length of the road is assumed to be changing with time so that the density of the vehicles on the ring may be varied without changing the number of cars. In the formulation below, the system is modeled as a dynamical system with input L and outputs h and v, the headway and velocity of the $(N-1)$-th vehicle in the platoon.

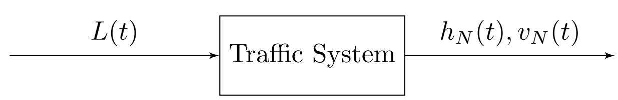

Figure 1: Open Loop System Diagram

Let s_i and v_i denote the position and velocity of the i-th vehicle respectively. We assume that the system obeys car-following dynamics, where for $0 < i < N$:

$$\dot{s_i} = v_i \tag{1}$$

$$\dot{v_i} = f(s_{i-1} - s_i, v_{i-1} - v_i, v_i) \tag{2}$$

and for $i = 0$:

$$\dot{s_0} = v_0 \tag{3}$$
$$\dot{v_0} = f(s_N - s_0 + L, v_N - v_0, v_0) \tag{4}$$

where f is some nonlinear function. For the remainder of this post, we assume that f is the function corresponding to the linear optimal velocity model:

$$f(h, \dot{h}, v) = \begin{cases} 0 & h < h_{\min} \\ \alpha \left(v_{\max} \frac{h - h_{\min}}{h_{\max} - h_{\min}} - v \right) + \beta \dot{h} & h_{\min} < h < h_{\max} \\ v_{\max} & h > h_{\max} \end{cases}$$

where $v_{\max}$, $h_{\min}$, $h_{\max}$ are fixed parameters of the system. For convenience we define the following constant

$$k = \frac{\alpha v_{\max}}{h_{\max} - h_{\min}}$$

Although the system above described is nonlinear, assuming that all the headways are in the saturation region $h_{\min} < h < h_{\max}$ for all time, it may be transformed into an affine-linear system.

Let $\mathbf{x} \in \mathbb{R}^{2N}$, $\mathbf{s} \in \mathbb{R}^N$, $\mathbf{v} \in \mathbb{R}^N$, where

$$\mathbf{s} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} s_0 \\ v_0 \\ \vdots \\ s_{N-1} \\ v_{N-1} \end{bmatrix}$$

Then the system described by equations (1)-(4) may be rewritten as

$$\dot{\mathbf{s}}(t) = \mathbf{A}_0 \mathbf{x}(t)$$

$$\dot{\mathbf{v}}(t) = \mathbf{A}_s \mathbf{s}(t) + \mathbf{A}_v \mathbf{v}(t)$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \mathbf{A}_s = \begin{bmatrix} -k & 0 & 0 & \cdots & 0 & k \\ k & -k & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & -k \end{bmatrix}$$

$$\mathbf{A}_v = \begin{bmatrix} -\beta - \alpha & 0 & 0 & \cdots & 0 & \beta \\ \beta & -\beta - \alpha & 0 & \cdots & 0 & 0 \\ 0 & \beta & -\beta - \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta & -\beta - \alpha \end{bmatrix}$$

The matrix $\mathbf{A}_1$ is found by interleaving the columns of $\mathbf{A}_s$ and $\mathbf{A}_v$. Next, interleaving the rows of $\mathbf{A}_0$ and $\mathbf{A}_1$ gives us the $\mathbf{A}$ matrix in the affine linear state space representation of the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}L(t) + \mathbf{x}_{\text{offset}} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}L(t)\end{aligned}$$

with $\mathbf{x}_{\text{offset}} = [0 \quad \cdots \quad -k]^\top$ and $\mathbf{B} = [0 \quad 0 \quad \cdots \quad 0 \quad k]^\top$. The output and feed-forward matrices are defined by

$$\mathbf{C} = \begin{bmatrix} -1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

In general, an N-dimensional affine linear dynamical system may be transformed into an $(N+1)$-dimensional linear system

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \widetilde{\mathbf{A}}\mathbf{z}(t) + \widetilde{\mathbf{B}}L(t) \\ \mathbf{y}(t) &= \widetilde{\mathbf{C}}\mathbf{z}(t) + \widetilde{\mathbf{D}}L(t)\end{aligned}$$

where $\mathbf{z}(t) = [1 \;\; \mathbf{x}(t)^\top]^\top$ and

$$\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{x}_{\text{offset}} & \mathbf{A} \end{bmatrix} \qquad \widetilde{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix}$$

$$\widetilde{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \end{bmatrix} \qquad \widetilde{\mathbf{D}} = \mathbf{D}$$

Finally we discretize the linear system assuming a zero-order hold for the input, defining $\mathbf{C}_d = \widetilde{\mathbf{C}}$ and $\mathbf{D}_d = \widetilde{\mathbf{D}}$ and

$$\mathbf{A}_d = \exp\{\widetilde{\mathbf{A}}T_{\text{sample}}\} \qquad \mathbf{B}_d = \left(\int_0^{T_{\text{sample}}} \exp\{\widetilde{\mathbf{A}}\tau\} d\tau \right) \widetilde{\mathbf{B}}$$

where the final state space representation of the discrete time linear dynamical system is:

$$\begin{aligned}\mathbf{z}[k+1] &= \mathbf{A}_d\mathbf{z}[k] + \mathbf{B}_d L[k] \\ \mathbf{y}[k] &= \mathbf{C}_d\mathbf{z}[k] + \mathbf{D}_d L[k]\end{aligned}$$

System Inversion

Assume that we are given a discrete time linear dynamical system:

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}L[k] \\ \mathbf{v}_N[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}L[k]\end{aligned}$$

Our goal is to model a system taking input $y[k]$ and return $\widetilde{L}[k]$ such that $\widetilde{L}[k] \approx L[k]$. This system will be referred to as the inverse system to the one above.

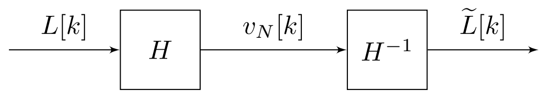

Figure 2: Cascasde with Inverse System

Assuming that the input to the system is unknown, cascading with the inverse system will allow us to approximate the input, which we can then use to infer the characteristic velocity. We employ the Massey-Sain Algorithm, published in IEEE Transactions on Automatic Control, vol. 14, 1969. Observe that

$$\begin{aligned}\mathbf{v}_N[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}L[k] \\ \mathbf{v}_N[k+1] &= \mathbf{C}\mathbf{A}\mathbf{x}[k] + \mathbf{C}\mathbf{B}L[k] + \mathbf{D}L[k]\end{aligned}$$

Continuing this iteration, we get

$$\mathbf{Y}_{k,M} = \mathcal{O}_M \mathbf{x}[k] + \mathbf{M}_M L_{k,M}$$

where

$$\mathbf{Y}_{k,M} = \begin{bmatrix} y[k] \\ y[k+1] \\ \vdots \\ y[k+M] \end{bmatrix} \qquad \mathcal{O}_M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^M \end{bmatrix}$$

$$\mathbf{M}_M = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{M-1}B & CA^{M-2} & CA^{M-3} & \cdots & D \end{bmatrix} \qquad L_{k,M} = \begin{bmatrix} L[k] \\ L[k+1] \\ \vdots \\ L[k+M] \end{bmatrix}$$

The main result is summarized in the theorems below:

Theorem 0.1. *The system H described above has inverse with lag M if and only if*

$$\mathit{rank}(\mathbf{M}_M) = \mathit{rank}(\mathbf{M}_{M-1}) + m$$

where m is the delay of the forward system.

Theorem 0.2. *If the system H has a delay with lag M there exists a matrix $\mathcal{K}$ such that*

$$\mathcal{K}\mathbf{Y}_{k,M} = [\mathbf{I}_M \quad \mathbf{0}]$$

The state-space representation of the inverse system is given by

$$\begin{aligned} x[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}L[k] \\ &= (\mathbf{A} - \mathbf{B}\mathcal{K}\mathcal{O}_M)x[k] + \mathbf{B}\mathcal{K}\mathbf{Y}_{k,M} \\ L[k] &= -\mathcal{K}\mathcal{O}_M x[k] + \mathcal{K}\mathbf{Y}_{k,M} \end{aligned}$$

Applications and Future Work

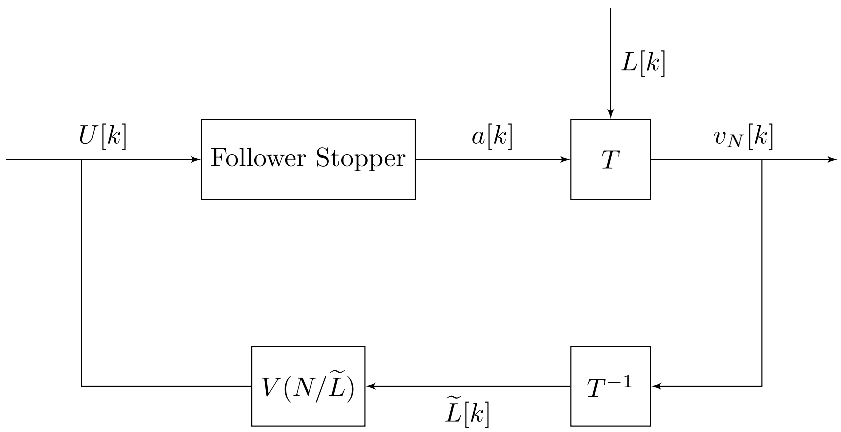

Figure 3: Closed Loop System

The following feedback system is proposed to mitigate traffic. Using the inverse system above, the unknown input L to the traffic system is approximated. The characteristic velocity is inferred from this value, which is then fed in as an input to the FollowerStopper. Here, T is the traffic system computed in Part I and T^{-1} is the inverse computed in Part II.

We can generalized the system engineering above below

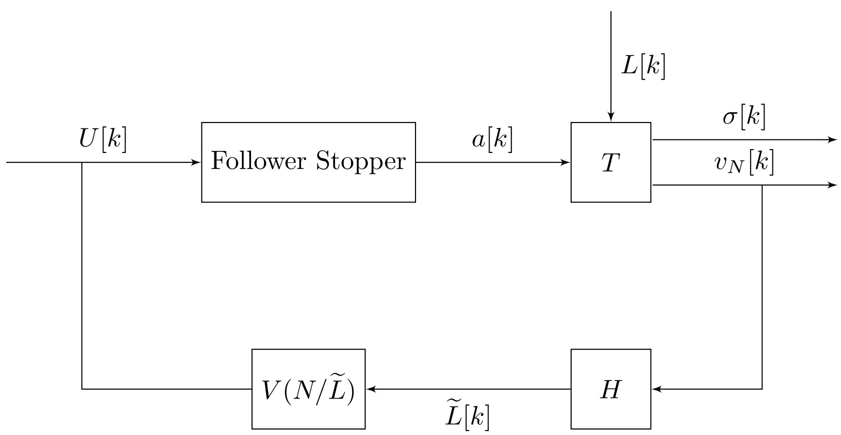

Figure 4: Closed Loop System

where H is a system that approximates L using measurements σ. In the case above, $\sigma = v_N$ and H is the inverse system. In the future we explore approaches where σ represents other measurements extracted from the traffic system. In particular we may employ data driven approach, where H is found using system identification. In the next post we consider these approaches as well as comparing results.