Plane Stress

Displacement function: u & v in-plane displacement

$$u = \sum_i \sum_j a_{ij} \xi^i \eta^j$$

$$v = \sum_i \sum_j b_{ij} \xi^i \eta^j$$

Strain Energy Calculation:

$$U = \frac{Eh}{2(1-\nu^2)} \int_{-1}^{1}\int_{-1}^{1} \Big[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\nu \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{(1-\nu)}{2}\left(\frac{\partial u}{\partial y}\right)^2$$

$$+ (1-\nu)\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{(1-\nu)}{2}\left(\frac{\partial v}{\partial x}\right)^2 \Big] |J| \, d\xi \, d\eta$$

$$|J| = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial \xi} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\xi^{i-1}\eta^j$$

$$\frac{\partial u}{\partial \eta} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} a_{ij}\xi^{i}\eta^{j-1}$$

$$\frac{\partial v}{\partial \xi} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} b_{ij}\xi^{i-1}\eta^j$$

$$\frac{\partial v}{\partial \eta} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} b_{ij}\xi^{i}\eta^{j-1}$$

Side Note:

Polynomial series: (multi-dimensional)

$$u(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}x^i y^j$$

$$= a_{00} + a_{01}y^1 + a_{02}y^2 + a_{03}y^3 + a_{04}y^4 + \cdots$$
$$+ a_{11}xy + a_{12}xy^2 + a_{13}xy^3 + a_{14}y^{4} + \cdots$$
$$+ a_{21}x^2y + a_{22}x^2y^2 + a_{23}x^2y^3 + a_{24}x^2y^4 + \cdots$$
$$+ \cdots$$

$$\frac{\partial u}{\partial x} = a_{11}y + a_{12}y^2 + a_{13}y^3 + a_{14}y^4 + \cdots$$
$$+ a_{21}xy + a_{22}xy^2 + a_{23}xy^3 + a_{24}xy^4 + \cdots$$
$$+ \cdots$$

$$\Rightarrow \frac{\partial u}{\partial x} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} a_{ij}x^{i-1}y^j$$

$$\Rightarrow \frac{\partial u}{\partial y} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} a_{ij}x^{i}y^{j-1}$$

To solve $\frac{\partial \xi}{\partial x}$, $\frac{\partial \xi}{\partial y}$, $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$

$$x = C_1 \xi^2 \eta + C_2 \xi^2 + C_3 \xi + C_4 \xi \eta^2 + C_5 \eta^2 + C_6 \eta + C_7 \xi \eta + C_8$$

$$y = D_1 \xi^2 \eta + D_2 \xi^2 + D_3 \xi + D_4 \xi \eta^2 + D_5 \eta^2 + D_6 \eta + D_7 \xi \eta + D_8$$

Using Implicit Differentiation:

take $\frac{\partial}{\partial x}$ on both sides, consider $\xi(x,y)$, $\eta(x,y)$

$$\left\{\begin{aligned} 1 = {} & C_1 \cdot 2\xi \frac{\partial \xi}{\partial x} \eta + C_1 \xi^2 \frac{\partial \eta}{\partial x} + C_2 \cdot 2\xi \frac{\partial \xi}{\partial x} + C_3 \frac{\partial \xi}{\partial x} \\ & + C_4 \frac{\partial \xi}{\partial x} \eta^2 + C_4 \xi \cdot 2\eta \frac{\partial \eta}{\partial x} + C_5 \cdot 2\eta \frac{\partial \eta}{\partial x} + C_6 \frac{\partial \eta}{\partial x} + C_7 \frac{\partial \xi}{\partial x} \eta + C_7 \xi \frac{\partial \eta}{\partial x} \\ 0 = {} & D_1 \cdot 2\xi \frac{\partial \xi}{\partial x} \eta + D_1 \xi^2 \frac{\partial \eta}{\partial x} + D_2 \cdot 2\xi \frac{\partial \xi}{\partial x} + D_3 \frac{\partial \xi}{\partial x} \\ & + D_4 \frac{\partial \xi}{\partial x} \eta^2 + D_4 \xi \cdot 2\eta \frac{\partial \eta}{\partial x} + D_5 \cdot 2\eta \frac{\partial \eta}{\partial x} + D_6 \frac{\partial \eta}{\partial x} + D_7 \frac{\partial \xi}{\partial x} \eta + D_7 \xi \frac{\partial \eta}{\partial x} \end{aligned}\right.$$

→ 2 equations, 2 variables $\frac{\partial \xi}{\partial x}$ & $\frac{\partial \eta}{\partial x}$, solvable

Same for $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$, solvable

Work of Applied Load :

$$T = \int_S \frac{1}{|t|^2}\left[(-f_n t_y + f_t t_x)u + (f_n t_x + f_t t_y)v\right]\cdot\left[\left(t_x\frac{\partial x}{\partial \xi} + t_y\frac{\partial y}{\partial \xi}\right)d\xi + \left(t_x\frac{\partial x}{\partial \eta} + t_y\frac{\partial y}{\partial \eta}\right)d\eta\right]$$

on one edge

$$T = \sum_{i=1}^{4} T_i \quad ?$$

For this example

need mappings for edges

dependents on edges

— For side $\underline{\xi = 1}$, $f_n = \sigma h$, $f_t = 0$, $\underline{t_x = t_y = 1}$, $|t| = \sqrt{2}$

$$T = \int_S \frac{1}{|t|^2}\left[(-f_n t_y + f_t t_x)u + (f_n t_x + f_t t_y)v\right]\cdot\left[\left(t_x\frac{\partial x}{\partial \xi} + t_y\frac{\partial y}{\partial \xi}\right)d\xi + \left(t_x\frac{\partial x}{\partial \eta} + t_y\frac{\partial y}{\partial \eta}\right)d\eta\right]$$

$$= \int_S \frac{1}{2}\left[-\sigma h\cdot(1)\cdot u + \sigma h(1)\cdot v\right]\cdot\left[\left((1)\cdot 4 + (1)\cdot 0\right)d\xi + \left((1)\cdot 2\sqrt{2} + (1)\cdot 2\sqrt{2}\right)d\eta\right]$$

$$= \int_{-1}^{1} \frac{h}{2}\left(-\sigma\cdot u + \sigma\cdot v\right)_{\xi=1}\cdot 4\sqrt{2}\, d\eta$$

$$= 2\sqrt{2}\,\sigma h \int_{-1}^{1} (-u + v)\, d\eta$$

$$= 2\sqrt{2}\,\sigma h \int_{-1}^{1} (b_{ij} - a_{ij})\,\eta^i\, d\eta \quad ?$$

Lagrange Multiplier:

In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints (i.e., subject to the condition that one or more equations have to be satisfied exactly by the chosen values of the variables).

Method: in order to find the max or min of a function $f(x)$ subject to the equality constraints $g(x) = 0$

$$\mathcal{L}(x, \lambda) = f(x) + \lambda g(x)$$

and find the stationary points of $\mathcal{L}$ considered as a function of x and the Lagrange multiplier λ_i.

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \underbrace{g(x, y)}_{\text{constraint } g(x,y) = 0}$$

Gradient:

$$\nabla_{x,y,\lambda} \mathcal{L}(x, y, \lambda) = \left(\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial y}, \frac{\partial \mathcal{L}}{\partial \lambda} \right) = 0$$

In this case:

$$f(a_{ij}, b_{ij}) = U(a_{ij}, b_{ij}) - T(a_{ij}, b_{ij})$$

$$g(a_{ij}, b_{ij}) = R_{m \times 1} \qquad m = ? : \text{dependents on \# of constraints}$$

$$\lambda = 1 \times m \text{ Matrix}$$

$$\frac{\partial f}{\partial a_{ij}} = \lambda \frac{\partial g}{\partial a_{ij}} \qquad \frac{\partial f}{\partial b_{ij}} = \lambda \frac{\partial g}{\partial b_{ij}} \qquad g(a_{ij}, b_{ij}) = 0$$

To be more general:

parameter a has a dimension of $i \times j$

parameter b has a dimension of $p \times q$

So, a_{ij} has $(i+1)(j+1)$ elements

b_{pq} has $(p+1)(q+1)$ elements

λ_m has m elements

$\uparrow$ m dependents on how many edges have been constrained

Therefore, totally $[(i+1)(j+1)+(p+1)(q+1)+m]$ unknowns to be solved.

Equation Group #1: $\dfrac{\partial f}{\partial a_{ij}} = \lambda_m \dfrac{\partial g}{\partial a_{ij}}$

has $(i+1)(j+1)$ equations

Equation Group #2: $\dfrac{\partial f}{\partial b_{pq}} = \lambda_m \dfrac{\partial g}{\partial b_{pq}}$

has $(p+1)(q+1)$ equations

Equation Group #3: $g(a_{ij}, b_{pq}) = 0$

has m equations

Therefore, totally $[(i+1)(j+1)+(p+1)(q+1)+m]$ equations to use.

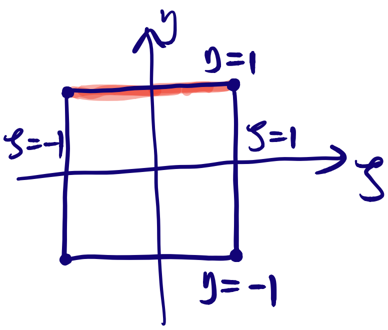
Should be solvable !!!

Constraints R

First constrain one edge, set displacement (u and v) on the side $\eta=1$ to zero:

$$u(\xi,1)=\sum_i\sum_j a_{ij}\xi^i(1)^j=\sum_i\sum_j a_{ij}\xi^i=0$$

$$v(\xi,1)=\sum_i\sum_j b_{ij}\xi^i(1)^j=\sum_i\sum_j b_{ij}\xi^i=0$$



$$u(\xi,1)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\xi^i=0$$

$$\xrightarrow{i=0} = a_{00}\xi^0+a_{01}\xi^0+a_{02}\xi^0+a_{03}\xi^0+a_{04}\xi^0+\dots\dots$$

$$\xrightarrow{i=1} + a_{10}\xi^1+a_{11}\xi^1+a_{12}\xi^1+a_{13}\xi^1+a_{14}\xi^1+\dots\dots$$

$$\xrightarrow{i=2} + a_{20}\xi^2+a_{21}\xi^2+a_{22}\xi^2+a_{23}\xi^2+a_{24}\xi^2+\dots$$

$$+\dots\dots$$

$$\xrightarrow{i=i} + a_{i0}\xi^i+a_{i1}\xi^i+a_{i2}\xi^i+a_{i3}\xi^i+a_{i4}\xi^i+\dots$$

$$=[a_{00}+a_{01}+a_{02}+a_{03}+a_{04}+\dots]\xi^0$$

$$+[a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+\dots]\xi^1$$

$$+[a_{20}+a_{21}+a_{22}+a_{23}+a_{24}+\dots]\xi^2$$

$$+\dots\dots$$

$$+[a_{i0}+a_{i1}+a_{i2}+a_{i3}+a_{i4}+\dots]\xi^i$$

$\Longrightarrow$ Constrain Equations:

$$R\left\{\begin{array}{l} a_{00}+a_{01}+a_{02}+a_{03}+a_{04}+\dots+a_{0j}=0 \\ a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+\dots+a_{1j}=0 \\ a_{20}+a_{21}+a_{22}+a_{23}+a_{24}+\dots+a_{2j}=0 \\ \dots\dots \\ a_{i0}+a_{i1}+a_{i2}+a_{i3}+a_{i4}+\dots+a_{ij}=0 \end{array}\right\}\ (i+1)\text{ equations}$$

$\Rightarrow$ $(i+1)$ constraint equations for a_{ij}:

$$a_{i0}+a_{i1}+a_{i2}+\dots+a_{ij}=0$$

$(i+1)$ constraint equations for b_{ij}:

$$b_{i0}+b_{i1}+b_{i2}+\dots+b_{ij}=0$$

Apply Minimum Potential Energy

$$\Pi = U - T$$

$$d\Pi = \left(\frac{\partial U}{\partial a_{ij}} - \frac{\partial T}{\partial a_{ij}}\right) da_{ij} + \left(\frac{\partial U}{\partial b_{ij}} - \frac{\partial T}{\partial b_{ij}}\right) db_{ij} = 0$$

$[K] \sim \frac{\partial U}{\partial a_{ij}}$ & $\frac{\partial U}{\partial b_{ij}}$ $\leftrightarrow$ Minimization of U

$[F] \sim \frac{\partial T}{\partial a_{ij}}$ & $\frac{\partial T}{\partial b_{ij}}$ $\leftrightarrow$ Minimization of T

$\{x\} \sim a_{ij}$ & b_{ij}

$$\bar{\Pi} = U - T + \lambda_i R_i$$

$$d\bar{\Pi} = \left(\frac{\partial U}{\partial a_{ij}} - \frac{\partial T}{\partial a_{ij}} + \frac{\partial \lambda_i R_i}{\partial a_{ij}}\right) da_{ij} + \left(\frac{\partial U}{\partial b_{ij}} - \frac{\partial T}{\partial b_{ij}} + \frac{\partial \lambda_i R_i}{\partial b_{ij}}\right) db_{ij}$$

$$+ \left(\cancelto{0}{\frac{\partial U}{\partial \lambda_i}} - \cancelto{0}{\frac{\partial T}{\partial \lambda_i}} + \frac{\partial \lambda_i R_i}{\partial \lambda_i}\right) d\lambda_i = 0$$

$$\begin{bmatrix} K & \mathbb{R}^T \\ \mathbb{R} & 0_{1\times 1} \end{bmatrix} \begin{Bmatrix} x \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \Rightarrow \text{eigenvalue problem}$$

$\rightarrow$ coefficient of u and v : a_{ij} & b_{ij}

$\rightarrow$ stress & strain throughout the plate.

Buckling (Bending)

Displacement Function : out-of-plane displacement ω

$$\omega = \sum_i \sum_j c_{ij} \xi^i \eta^j$$

Strain Energy

$$U = \frac{D}{2}\int_A \left\{ \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 \omega}{\partial x^2}\frac{\partial^2 \omega}{\partial y^2} - \left(\frac{\partial^2 \omega}{\partial x \partial y}\right)^2\right]\right\} dA$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$\Downarrow$ in (x,y) domain

$$U = \frac{D}{2}\int_{-1}^{1}\int_{-1}^{1} \left\{ \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 \omega}{\partial x^2}\frac{\partial^2 \omega}{\partial y^2} - \left(\frac{\partial^2 \omega}{\partial x \partial y}\right)^2\right]\right\} \cdot |J|\, d\xi\, d\eta$$

$\Downarrow$ in (ξ,η) domain

$$|J| = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \omega}{\partial y} = \frac{\partial \omega}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{\partial^2 \omega}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 \omega}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial \omega}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial \omega}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + 2\frac{\partial^2 \omega}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 \omega}{\partial y^2} = \frac{\partial^2 \omega}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 \omega}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial \omega}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial \omega}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + 2\frac{\partial^2 \omega}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 \omega}{\partial x \partial y} = \frac{\partial^2 \omega}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 \omega}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 \omega}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \omega}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial \omega}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$

Work of External Loads

$$T = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left\{N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}\cdot\left|J(\xi,\eta)\right| d\xi\, d\eta$$

— T is evaluated in (ξ, η) domain using Gaussian integration, with N_x, N_y and N_{xy} defined at each Gaussian point.

$$\begin{cases} N_x = \sigma_{xx} h \\ N_y = \sigma_{yy} h \\ N_{xy} = \sigma_{xy} h \end{cases} \Leftarrow$$ with the stresses at the Gauss point yielding N_x, N_y, N_{xy}.

$$[\sigma] = [G]\{\varepsilon\}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \left\{\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}^T$$

$$[G] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

$$[\sigma] = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \Leftarrow$$ Results from plane stress

— $$|J| = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y}$$

Apply Minimum Potential Energy

$$\overline{\Pi} = U - T + \lambda_i R_i$$

Minimization of $\overline{\Pi}$ yields the critical Loads :

$$\frac{\partial \overline{\Pi}}{\partial c_{ij}} = 0 \quad \text{and} \quad \frac{\partial \overline{\Pi}}{\partial \lambda_i} = 0$$

This gives the eigenvalue problem :

$$\left(\begin{bmatrix} \frac{\partial U}{\partial c_{ij}} & R_i^T \\ R_i & 0 \end{bmatrix} - P \begin{bmatrix} \frac{\partial T}{\partial c_{ij}} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} c_{ij} \\ \lambda_i \end{Bmatrix} = 0$$

P : critical load factor