Setup:
Coordinates:
E: an inertial / fixed frame
B: a body-fixed frame, of which the positive x direction aligns with the direction of the propeller arm relative to the center of mass of the body. The origin lies at the center of mass of the whole flyer.
C: a control frame dependent on the
State:
ω_{BE}^{B}: the angular velocity of the body relative to the inertial frame expressed in the body frame.
I_B^B: the moment of inertia of the body (without the propeller) with respect to the body's center of mass
I_P^B: the moment of inertia of the propeller with respect to the propeller's center of mass
e_P^B: propeller force direction in the body frame
τ_d^B: air frame drag torque
f_P: thrust of the propeller
τ_P: torque of the propeller
u: the output thrust produced by the attitude controller
$\Delta\tau$: time constant of the motor
Calculation:
The linearization is performed by assembling and controlling a state vector *s*.
Assume angular velocity, position, linear velocity, and the current thrust can be obtained through either IMU or outside localization system.

From Euler's Second Law, the angular acceleration of the body in the body frame $\dot{\omega}_{BE}^{B}$ can be estimated.

$$I_B^B\dot{\omega}_{BE}^B + \omega_{BE}^B \times (I_B^B\omega_{BE}^B) = r_P^B \times e_P^B f_B + e_P^B\tau_P + \tau_d^B$$

$$I_B^B\dot{\omega}_{BE}^B = r_P^B \times e_P^B f_B + e_P^B\tau_P + \tau_d^B - \omega_{BE}^B \times (I_B^B\omega_{BE}^B) \qquad (1)$$

The angular acceleration of the propeller in the body frame $\dot{\omega}_{PE}^{B}$ can be calculated from the derivative of the input thrust:

$$\dot{f}_P = (f_{position} + u - f_P)/\Delta\tau \qquad (2)$$

Total thrust *f* produced by the propeller has the following relation with the angular velocity of the propeller relatively to the earth in the body frame ω_{PE}^{B}.

$$f_P = \kappa_f(\boldsymbol{\omega}_{PE}^B \cdot \boldsymbol{e}_P^B)|\boldsymbol{\omega}_{PE}^B \cdot \boldsymbol{e}_P^B| \qquad (3)$$

And ω_{PE}^{B} can be decomposed into the angular velocity of the propeller relative to the body and the angular velocity of the body relative to the earth.

$$\boldsymbol{\omega}_{PE}^B = \boldsymbol{\omega}_{PB}^B + \boldsymbol{\omega}_{BE}^B \qquad (4)$$

In order to implement the simulation, an assumption is made: $\omega_{PB}^B \gg \omega_{BE}^B$. Therefore,

$$\omega_{PE}^B \approx \omega_{PB}^B$$

Since $\omega_{PB}^B = [0; 0; \Omega]$, where Ω is the scalar rotation speed of the propeller, Then equation (1) can be solved.

Next, the control frame C is defined such as the following condition is satisfied, where C^{CB} is the transformation matrix from B to C

$$n^C = C^{CB} n^B = [0; 0; 1]$$

where n^B is the unit vector in the direction of the average angular velocity of the body in the body frame when the flyer is in hover state:

$$n^B = \pm \frac{\overline{\omega_{BE}^B}}{\left|\omega_{BE}^B\right|} \quad (10)$$

Angular velocity of the body in the control frame:

$$\omega_{BE}^C = C^{CB} \omega_{BE}^B \qquad (11)$$

The derivative of the desired acceleration in the body frame n_{des}^B is calculated by

$$\dot{n}_{des}^B = -\omega_{BE}^B \times n_{des}^B \qquad (12)$$

since n_{des}^B generated by position control is regarded as constant for the attitude control.
In control frame C,

$$\dot{n}_{des}^C = -\omega_{BE}^C \times n_{des}^C \qquad (13)$$

Define state *s*

Let $n_{des}^C = [\eta_1; \eta_2; \eta_3]$ and $\omega_{BE}^C = [\alpha_1; \alpha_2; \alpha_3]$

State *s* is defined as such:

$$s = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ f_P \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pm\|\overline{\omega}_{BE}^B\| \\ \overline{f_P} \end{bmatrix}$$

Notice only x and y components of n_{des}^C are in *s* since n_{des}^C is a unit vector and the third component can be calculated from the first 2.

$\dot{s}$ can be calculated from equations (1) (2) and (13). Define a matrix function

$$\dot{s} = f(s, u)$$

Then the linearized form is

$$\dot{s} = As + Bu$$

where attitude thrust output *u* has the form $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u \end{bmatrix}$.

The model is linearized around the hover solution, where

$$s_{hover} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad u_{hover} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{f}_P \end{bmatrix}$$

Two matrices *A* and *B* are obtained by

$$A = Jacobian\left(\frac{\partial f}{\partial s}\right)_{hover}$$

$$B = Jacobian\left(\frac{\partial f}{\partial u}\right)_{hover}$$

References:

Zhang, W., Mueller, M. W., & Dandrea, R. (2016). A controllable flying vehicle with a single moving part. *2016 IEEE International Conference on Robotics and Automation (ICRA)*. doi:10.1109/icra.2016.7487499